Topic: • Divergence Theorem

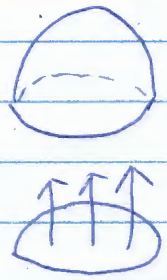
Defn: (Outward Flux)

- Let S be a piecewise smooth surface.
  Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.
  The outward flux of $\vec{F}$ across S is given by
  $\iint_S \vec{F} \cdot \vec{n} \, dA$, where $\vec{n}$ is the unit outward normal field on S.

Thm: (Divergence Theorem)

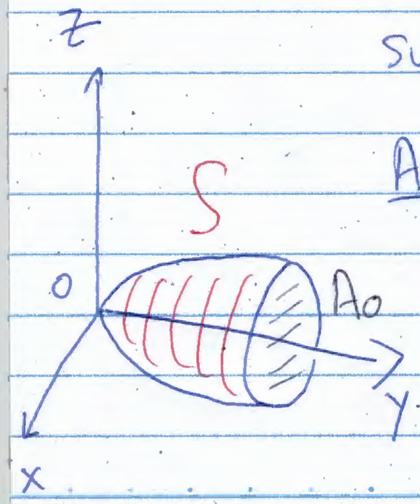
- Further suppose that S encloses a simply connected region D.
  Then $\iint_S \vec{F} \cdot \vec{n} \, dA = \iiint_D \nabla \cdot \vec{F} \, dV$

Remark: A surface may not be closed in general.



To apply divergence theorem, we need to add extra surface to make it close.

Example: 1) Find $\iint_S (0, y, -z) \cdot \vec{n} \, dA$, where S is the surface given by the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$.



Ans: Add $A_0 = \{(x, 1, z) \mid x^2 + z^2 \le 1\}$ to make it closed.

By divergence thm,

$$\iint_{S \cup A_0} (0, y, -z) \cdot n \, dA = \iiint_{S \cup A_0} \nabla \cdot (0, y, -z) \, dV = 0$$

On A_0, $n=(0,1,0)$

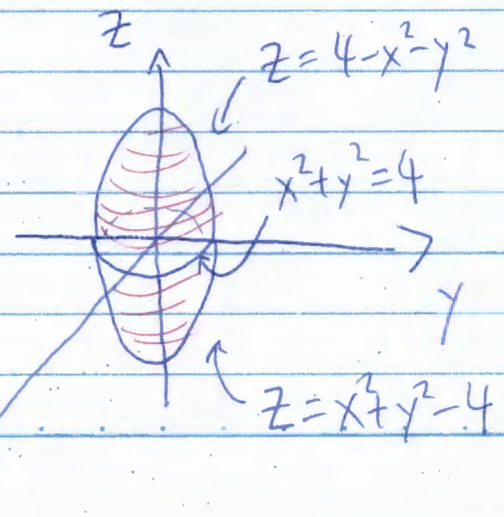
$$\Rightarrow \iint_{A_0} (0,y,-z)\cdot n\,dA = \iint_{A_0} (0,1,z)\cdot(0,1,0)\,dA$$
$$= \iint_{A_0} dA$$
$$= \pi$$

$$\therefore \iint_S (0,y,-z)\cdot n\,dA = 0 - \pi = -\pi$$

2) Let S be the union of truncated paraboloids $z=4-x^2-y^2$, $0\le z\le 4$ and $z=x^2+y^2-4$, $-4\le z\le 0$. Find $\iint_S (x+y^2+\sin z,\ x+y^2+\cos z,\ \cos x+\sin y+z)\cdot n\,dA$.

Ans: $\iint_S (x^2+y^2+\sin z,\ x+y^2+\cos z,\ \cos x+\sin y+z)\cdot n\,dA$

$$\overset{\text{div.}}{\underset{\text{thm.}}{=}} \iiint_D (2+2y)\,dV$$
$$= \int_0^{2\pi}\int_0^2\int_{r^2-4}^{4-r^2} (2+2r\sin\theta)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (2+2r\sin\theta)(r)(8-2r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (16r-4r^3+16r^2\sin\theta-4r^4\sin\theta)\,dr\,d\theta$$
$$= 2\pi\int_0^2 (16r-4r^3)\,dr$$
$$= 32\pi$$



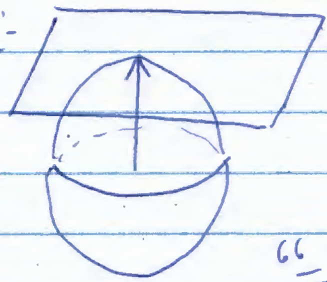
3) Let S be a smooth closed surface enclosing a region $R \subset \mathbb{R}^3$.
Show that
$$\mathrm{vol}(R) = \frac{1}{3}\iint_S \vec{r}\cdot\vec{n}\,dS,$$
where $\vec{r}(x,y,z) = (x,y,z)$ is the position vector of the points on S.

Ans: $\displaystyle \frac{1}{3}\iint_S \vec{r}\cdot\vec{n}\,dS \overset{\text{div.}}{\underset{\text{thm.}}{=}} \frac{1}{3}\iiint_R \nabla\cdot\vec{r}\,dV = \iiint_R dV = \mathrm{vol}(R)$ //

4) Let $\vec{F}(x,y,z) = \left(\frac{2x-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{x+2y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{2z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$ be a vector field on $\mathbb{R}^3\setminus\{0\}$.

(a) Find $\iint_S \vec{F}\cdot\vec{n}\,dA$, where $S = \{(x,y,z) \mid x^2+y^2+z^2 = R^2\}$.

Ans: Fact:-



Note that on sphere, outward normal "=" position vector

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_S \left(\frac{2x-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{x+2y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{2z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\cdot\frac{(x,y,z)}{R}\,dA$$
$$= \frac{1}{R^4}\iint_S (2x-y, x+2y, 2z)\cdot(x,y,z)\,dA$$
$$= \frac{1}{R^4}\iint_S 2(x^2+y^2+z^2)\,dA$$
$$= \frac{2}{R^2}\iint_S dA$$

$$= \frac{2}{R^2}(4\pi R^2)$$

$$= 8\pi$$

(b) Show that $\nabla \cdot \vec{F} = 0$.

<u>Ans:</u> $\nabla \cdot \vec{F}$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \left(\begin{array}{l} \dfrac{(x^2+y^2+z^2)^{\frac{3}{2}}(2) - (2x-y)(\frac{3}{2})(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+z^2)^3} \\ + \dfrac{(x^2+y^2+z^2)^{\frac{3}{2}}(2) - (x+2y)(\frac{3}{2})(x^2+y^2+z^2)^{\frac{1}{2}}(2y)}{(x^2+y^2+z^2)^3} \\ + \dfrac{(x^2+y^2+z^2)^{\frac{3}{2}}(2) - (2z)(\frac{3}{2})(x^2+y^2+z^2)^{\frac{1}{2}}(2z)}{(x^2+y^2+z^2)^3} \end{array} \right)$$

$$= \frac{6(x^2+y^2+z^2)^{\frac{3}{2}} - (6x^2-3xy+3xy+6y^2+6z^2)(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$= \frac{6(x^2+y^2+z^2)^{\frac{3}{2}} - 6(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$= 0$$

(c) Find $\iint_{\partial V} \vec{F} \cdot \vec{n}\, dA$, where ∂V is the boundary of:

(i) $V = \{(x,y,z) \mid 1 \le z \le 7 - x^2 - y^4\}$

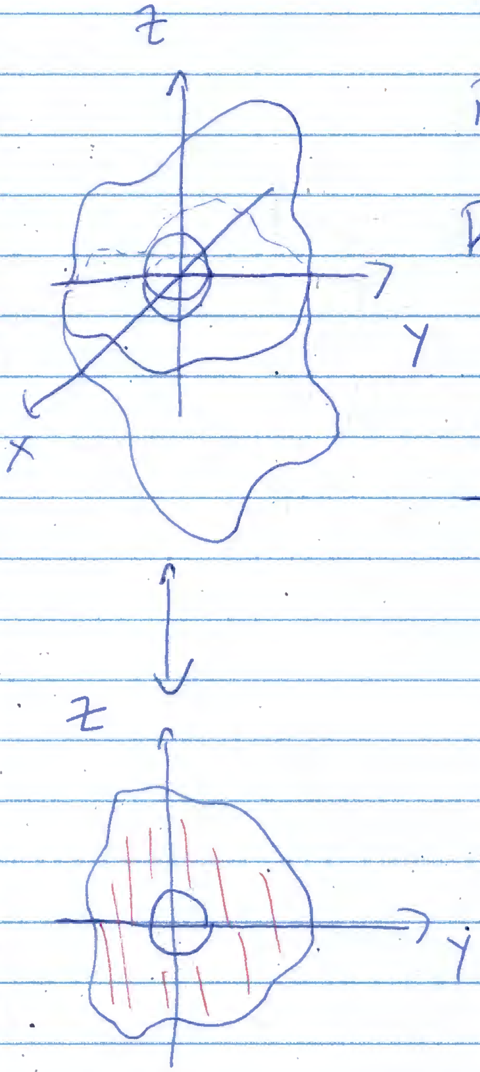
(ii) $V = \{(x,y,z) \mid -1 \le z \le 7 - x^2 - y^4\}$

Ans: (i) As $0 \notin V$, by divergence thm,

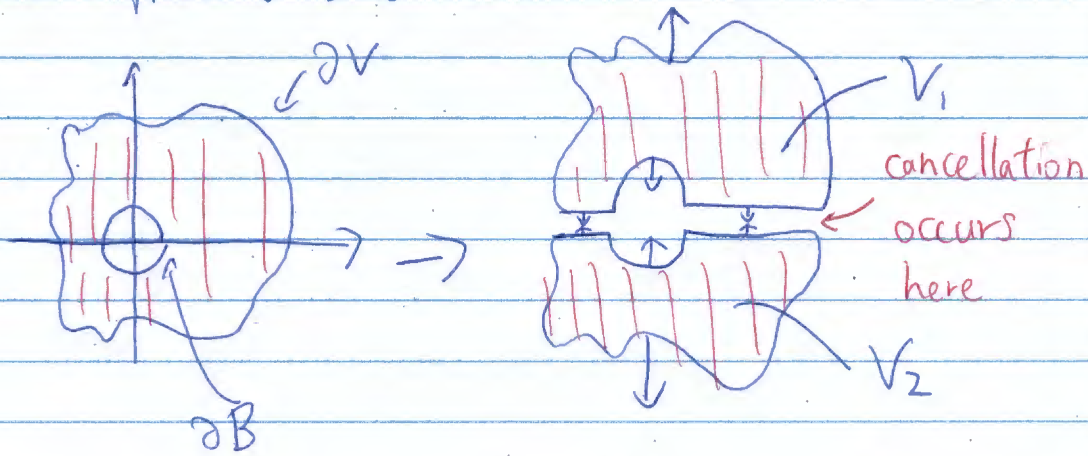
$$\iint_{\partial V} \vec{F} \cdot \vec{n}\, dA = \iint_{V} \nabla \cdot \vec{F}\, dA = 0 \quad \text{(by (b))}.$$

(ii) As $0 \in V$, we cannot apply div. thm directly.

Put a sufficiently small ball B centred at origin inside V.



Decompose $V \setminus B$ as follow:



Note that by div. thm, $\iint_{\partial V_1} \vec{F} \cdot \vec{n}\, dA = \iint_{\partial V_2} \vec{F} \cdot \vec{n}\, dA = 0$

$$\Rightarrow \iint_{\partial (V \setminus B)} \vec{F} \cdot \vec{n}\, dA = \iint_{\partial V_1} \vec{F} \cdot \vec{n}\, dA + \iint_{\partial V_2} \vec{F} \cdot \vec{n}\, dA = 0$$

$$\Rightarrow \iint_{\partial V} \vec{F} \cdot \vec{n}\, dA - \iint_{\partial B} \vec{F} \cdot \vec{n}\, dA = 0$$

$$\Rightarrow \iint_{\partial V} \vec{F} \cdot \vec{n}\, dA = \iint_{\partial B} \vec{F} \cdot \vec{n}\, dA = 8\pi \quad \text{(by (a))}$$

(This question is taken from the past paper of MATH2020B, 08-09)